Introduction to Noncommutative Geometry™

Sets. The category **Set** is <u>monoidal</u> with respect to the cartesian product $\mathbf{Set} \times \mathbf{Set} \xrightarrow{(-)\times(-)} \mathbf{Set}$ with the singleton $\bullet$ as a terminal object being the monoidal unit.

Proposition 1. Every set admits a unique <u>comonoid</u> structure in the monoidal category $(\mathbf{Set}, \times, \bullet)$

Proof. The comonoid structure is a pair of morphisms $X \xrightarrow{\Delta} X \times X$, $X \xrightarrow{\varepsilon} \bullet$ such that the following diagrams commute

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X \times X \arrow[d, "X \times \Delta"] \\
X \times X \arrow[r, "\Delta \times X"'] & X \times X \times X
\end{tikzcd}$$

(coassociativity)

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[dr, "\mathrm{Id}"'] & X \times X \arrow[d, shift right, "\varepsilon \times X"'] \arrow[d, shift left, "X \times \varepsilon"] \\
 & X
\end{tikzcd}$$

(counitality)

Since $\bullet$ is a terminal object ε is unique.

Every Δ must be of the form $\Delta(x) = (l(x), r(x))$ for some maps $l, r : X \to X$. By counitality $l = r = \mathrm{Id}$.

Therefore Δ is <u>the diagonal map</u> $\Delta(x) = (x, x)$.

<u>Coassociativity</u>:

$$\begin{tikzcd} x \arrow[r, mapsto, "\Delta"] \arrow[d, mapsto, "\Delta"'] & (x,x) \arrow[d, mapsto, "X \times \Delta"] \\ (x,x) \arrow[r, mapsto, "\Delta \times X"'] & (x,x,x) \end{tikzcd}$$

<u>Counitality</u>:

$$\begin{tikzcd} x \arrow[r, mapsto, "\Delta"] \arrow[dr, mapsto, bend right, "\mathrm{Id}"'] & (x,x) \arrow[d, mapsto, shift right, "\varepsilon \times X"'] \arrow[d, mapsto, shift left, "X \times \varepsilon"] \\ & x \end{tikzcd}$$

$\square$

Coalgebras. The category of vector spaces over $\mathbb{C}$ form a monoidal category with respect to the tensor product $\mathbf{Vect} \times \mathbf{Vect} \xrightarrow{(-)\otimes(-)} \mathbf{Vect}$, and the monoidal unit $\mathbb{C}$.

Proposition 2. The construction of a vector space $\mathbb{C}X$ with a given basis X and the forgetting of the vector space structure satisfy

$$\mathbf{Vect}(\mathbb{C}X, V) = \mathbf{Set}(X, V) \qquad \text{(adjunction)}$$

and

$$\begin{array}{ccc} \mathbb{C}(X \times Y) & \xrightarrow{\cong} & \mathbb{C}X \otimes \mathbb{C}Y \\ (x,y) & \longmapsto & x \otimes y \end{array}, \qquad \mathbb{C}\bullet \xrightarrow{\cong} \mathbb{C}$$

(strong comonoidality)

Proof. Adjunction is the definition of the basis of a vector space. Strong comonoidality is the construction of the tensor product in terms of the bases. $\square$

Corollary 1. For any set X $\mathbb{C}X$ is a comonoid in **Vect** (aka <u>coalgebra</u>).

Def. A coalgebra structure is a pair of linear maps $C \xrightarrow{\Delta} C \otimes C$, $C \xrightarrow{\varepsilon} \mathbb{C}$ satisfying coassociativity and counitality. Coalgebras form a category **Coalg** with linear maps preserving the coalgebra structure.

Sweedler-Heyneman convention. It is convenient to write the comultiplication $\Delta : C \longrightarrow C \otimes C$ as $c \longmapsto c_{(1)} \otimes c_{(2)}$ even if the right hand side is not a simple tensor. Since every tensor is a sum of simple tensors, it is referred to as "summation sign suppressing".

Then once iterated comultiplications are written as

$$(\Delta \otimes C)\Delta(c) = c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}$$

$$(C \otimes \Delta)\Delta(c) = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}$$

Note that both indexings are strictly increasing to the right in the linear order in which

$$(i_1)(i_2)\dots(i_k) < (j_1)(j_2)\dots(j_l)$$

if $\quad i_1 < j_1$

or $\quad i_1 = j_1,\ i_2 < j_2$

or $\quad i_1 = j_1,\ i_2 = j_2,\ i_3 < j_3$

or $\quad i_1 = j_1,\ i_2 = j_2,\ i_3 = j_3,\ i_4 < j_4$ ... etc.

Coassociativity means that indexing of an arbitrary iterated comultiplications $(i\dots) < (\dots) < \dots$ can be unambiguously replaced by indexing by the standard linear order $(i) < (i+1) < \dots$ of the same length, e.g.

$$\big((1)(1) < (1)(2) < (2)\big) \equiv \big((1) < (2) < (3)\big) \equiv \big((1) < (2)(1) < (22)\big).$$

Counitality reads as $\varepsilon(c_{(1)})\, c_{(2)} = c = c_{(1)}\, \varepsilon(c_{(2)})$,

and applied to an iterated comultiplication as

$$\varepsilon(c_{(1)})\, c_{(2)} \otimes \cdots \otimes c_{(n+1)} = c_{(1)} \otimes \cdots \otimes c_{(n)}$$

$$= c_{(1)} \otimes \cdots \otimes c_{(i)}\, \varepsilon(c_{(i+1)}) \otimes \cdots \otimes c_{(n+1)}$$

for $i = 1, \ldots, n$.

Proposition 3. The functor $\mathbf{Set} \to \mathbf{Vect}$, $X \rightsquigarrow \mathbb{C}X$ determines an adjunction

$$\mathbf{Coalg}(\mathbb{C}X, C) = \mathbf{Set}(X, E(C))$$

where for any coalgebra C

$$E(C) := \{ c \in C \mid \Delta(c) = c \otimes c,\ \varepsilon(c) = 1 \}$$

is the <u>set of elements</u> of C.

Proof. A coalgebra map $\mathbb{C}X \to C$ is equivalent to a map $X \xrightarrow{\varphi} C$ such that the following diagrams commute.

$$
\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "\Delta"'] & C \arrow[d, "\Delta"] \\
X \times X \arrow[r, "\varphi \otimes \varphi"] & C \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "\varepsilon"'] & C \arrow[d, "\varepsilon"] \\
\bullet \arrow[r, "1"] & \mathbb{C}
\end{tikzcd}
$$

i.e.

$$
\begin{tikzcd}
x \arrow[r, mapsto] \arrow[dd, mapsto] & \varphi(x) \arrow[d, mapsto] \\
 & \Delta\varphi(x) \arrow[d, phantom, "\shortparallel"] \\
(x,x) \arrow[r, mapsto] & \varphi(x)\otimes\varphi(x)
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, mapsto] \arrow[dd, mapsto] & \varphi(x) \arrow[d, mapsto] \\
 & \varepsilon(\varphi(x)) \arrow[d, phantom, "\shortparallel"] \\
\bullet \arrow[r, mapsto] & 1
\end{tikzcd}
$$

which means that $\varphi(x) \in E(C)$. $\square$

Proposition 4. The natural map $X \to E(\mathbb{C}X)$ is a bijection.

Proof. The natural map provided by the latter adjunction is $X \longrightarrow E(\mathbb{C}X)$, $x \longmapsto x$. (On the right hand side x means the formal linear combination $1 \cdot x$.)
It is obviously injective. To show that it is surjective we write $c = \sum_i c_i x_i \in E(\mathbb{C}X)$. Applying Δ we obtain

$$\Delta(c) = \sum_i c_i \, x_i \otimes x_i$$

$$c \otimes c = \sum_{i,j} c_i c_j \, x_i \otimes x_j$$

hence by linear independence argument $\Delta(c) = c \otimes c$ reads as $c_i c_j = \delta_{ij} c_j$, and $\varepsilon(c) = 1$ as $\sum_i c_i = 1$

which means that (c_i) is a complete system of orthogonal idempotents in the field $\mathbb{C}$. Therefore $\exists i_0 \quad c_i = \delta_{i i_0}$, hence for $x := x_{i_0}$

$$c = x . \qquad \blacksquare$$

Remark 1. By category theory this means that the functor $X \rightsquigarrow \mathbb{C}X$ induces bijections

$$\mathbf{Set}(X,Y) \xrightarrow{\cong} \mathbf{Coalg}(\mathbb{C}X, \mathbb{C}Y)$$

natural in X and Y. In other words, **Set** can be regarded as a full subcategory of **Coalg** through the <u>linearization functor</u>.

Remark 2. All the above suggests to think of coalgebras as generalized sets.

For categorical minds: **Set** is equivalent to a full subcategory in **Coalg** of coalgebras C such that the canonical coalgebra map $\mathbb{C}E(C) \longrightarrow C$, $c \longmapsto c$ is an isomorphism, or equivalently, to a co-reflective co-localization of **Coalg** with respect to the idempotent comonad $\mathbb{C}E(-)$.

Cumulative structure of sets and coalgebras.

It is obvious that every set is the union of its finite subsets (in fact singletons alone suffice). The following theorem says how this property extends to coalgebras.

Theorem 1. Every coalgebra is the union of its finite dimensional subcoalgebras.

Proof. It is sufficient to prove that every element c of a coalgebra C is contained in a finite dimensional subcoalgebra $D \subset C$.

Let $c \in C$ and X be a basis of C as a vector space.

$$\Delta(c) = \sum_{x \in X} c_x \otimes x = \sum_{x' \in X} c_{x'} \otimes x'$$

$$(\Delta \otimes C)\Delta(c) = \sum_{x \in X} \Delta(c_x) \otimes x$$

$$(C \otimes \Delta)\Delta(c) = \sum_{x, x' \in X} x' \otimes c_{x'x} \otimes x = \sum_{x \in X} \Big(\sum_{x' \in X} x' \otimes c_{x'x}\Big) \otimes x$$

$$\Rightarrow \quad \Delta(c_x) = \sum_{x' \in X} x' \otimes c_{x'x}, \quad c = \sum_{x, x' \in X} \varepsilon(c_{x'}) c_{x'x} \varepsilon(x)$$

Now, let $D := \operatorname{span}\{c_{x'x} \mid x, x' \in X\}$. Then $\dim D < \infty$ and $c \in D$.

We want to show that D is a subcoalgebra, i.e. $\Delta(D) \subset D \otimes D \subset C \otimes C$. Now,

$$(C \otimes \Delta \otimes C)(\Delta \otimes C)\Delta(c) = \sum_{x, x' \in X} x' \otimes \Delta(c_{x'x}) \otimes x$$

$$(\Delta \otimes C \otimes C)(\Delta \otimes C)\Delta(c) = \sum_{x, x' \in X} \Delta(x') \otimes c_{x'x} \otimes x$$

$$\Rightarrow \quad \sum_{x'} x' \otimes \Delta(c_{x'x}) = \sum_{x' \in X} \Delta(x') \otimes c_{x'x} \in C \otimes C \otimes D$$

$\Rightarrow \quad \Delta(D) \subset C \otimes D$. Similarly, $\Delta(D) \subset D \otimes C$ hence

$$\Delta(D) \subset (C \otimes D) \cap (D \otimes C) = D \otimes D \subset C \otimes C. \quad \square$$

For categorical minds. $\mathbf{Set}$ and $\mathbf{Coalg}$ are equivalent to their ind-categories of their full subcategories of finite objects, i.e.

$$\mathbf{Set} = \text{ind-}\mathbf{Set}_{\text{fin.}},$$

$$\mathbf{Coalg} = \text{ind-}\mathbf{Coalg}_{\text{fin.dim.}}.$$

Cocommutativity. The cartesian product in **Set** and the tensor product in **Vect** are symmetric. This means the existence of an isomorphism

$$X \times Y \xrightarrow{\cong} Y \times X \qquad \text{or} \qquad V \otimes W \xrightarrow{\cong} W \otimes V$$
$$(x,y) \longmapsto (y,x) \qquad\qquad\qquad v \otimes w \longmapsto w \otimes v$$

natural in both factors and satisfying some natural <u>coherence conditions</u> when iterated (cf. axioms of symmetric monoidal categories).

In such categories one can distinguish <u>cocommutative comonoids</u>, defined as those whose comultiplication is symmetric.

- All sets are cocommutative comonoids in **Set**.

$$\begin{tikzcd} & X\times X \arrow[d] \\ X \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & \\ & X\times X \end{tikzcd} \qquad \begin{tikzcd} & (x,x) \arrow[d, mapsto] \\ x \arrow[ur, mapsto] \arrow[dr, mapsto] & \\ & (x,x) \end{tikzcd}$$

- Not all coalgebras are cocommutative comonoids in **Vect**,

Example 1. For every small category $\mathcal{C}$ with finite decomposition property (every morphism m has finitely many decompositions $m = m_1 m_2$) $\mathbb{C}\,\mathrm{Mor}(\mathcal{C})$ admits another coalgebra structure

$$\Delta(m) = \sum_{m_1 m_2 = m} m_1 \otimes m_2 \,, \qquad \varepsilon(m) = \begin{cases} 1 & m \text{ unit}, \\ 0 & \text{otherwise}. \end{cases}$$

For instance, let $\mathcal{C} = I \times I \overset{t}{\underset{s}{\rightrightarrows}} I$ be the groupoid of pairs over a finite set I, with the composition

$$(i,j)(j,k) = (i,k)$$

and units (i,i). Then

$$\Delta(i,k) = \sum_j (i,j) \otimes (j,k) \quad , \quad \varepsilon(i,k) = \delta_{ik}$$

If $i \neq k$ $\Delta(i,k)$ is not symmetric, hence the above coalgebra $\mathbb{C}\,\mathrm{Mor}(\mathcal{C})$ is not cocommutative.

- There are cocommutative coalgebras which are not isomorphic to those in the image of the linearization functor $\mathbb{C}-$: **Set** $\to$ **Coalg**.

Example 2. $C = \mathbb{C}\{x, v\}$ with the coalgebra structure

$$\Delta(x) = x \otimes x, \qquad \Delta(v) = x \otimes v + v \otimes x$$

$$\varepsilon(x) = 1, \qquad \varepsilon(v) = 0.$$

Then $E(C) = \{x\}$ and hence the natural coalgebra map $\mathbb{C}E(C) \longrightarrow C$ is not an isomorphism.